★ **Goal** Compute $\pi_\bigstar^{C_2} KR$ as Mackey functor.

KR = Atiyah KR theory. is a genuine C_2-spectrum.

note that $\pi_V^{C_2} KR(X) = [S^V \wedge X_+, KR]^{C_2}$ for X C_2-space.

Tool Slice SS (equivariant Postnikov tower)

$$E_2^{s,t} = \pi_{t-s}^{C_2} P_t KR \Rightarrow \pi_{t-s}^{C_2} KR.$$

FACT 1 By Bott periodicity, $KR^{p,q}(X) \cong KR^{p+2,q+1}(S^{2,1} \wedge X)$

So $\forall V \in R(G)$. $V = a + b\sigma$. ρ = regular rep $= 1+\sigma$

$$\pi_V^{C_2} KR = \pi_{a+b\sigma}^{C_2} KR = \pi_{a-b+b+b\sigma}^{C_2} KR$$

$$= \pi_{a-b+b\rho}^{C_2} KR$$

$$= \pi_{a-b}^{C_2} KR. \qquad a-b \in \mathbb{Z}.$$

That is, $\pi_\bigstar^{C_2} KR$ is determined by $\pi_*^{C_2} KR$. $*$ = $\mathbb{N}$-index.

No need to consider $RO(G)$-grading.

FACT 2 Recall that $KR(C_2 \times X) = KU(X)$

$KR(X) = KO(X)$ if C_2 acts trivially, i.e. $X^{C_2} = X$.

Thus $\pi_*^{C_2} KR(C_2/C_2) = \pi_* KO$

$\pi_*^{C_2} KR(C_2/e) = \pi_* KU$

FACT 3   Homotopy fixed point theorem

$$KR^{C_2} = KR^{hC_2}$$

(*) In fact. if X is equivariant htpy ring spectrum and $\widetilde{EG} \wedge X$ is contractible. $X \longrightarrow X^{EG_+}$ is then a weak equiv. and so

$$X^G \longrightarrow (X^{EG_+})^G = X^{hG} \quad \text{equiv.}$$

- We check that $\widetilde{EC_2} \wedge KR \simeq *$.

In the isotropy separation sequence

$$E\mathcal{F}_+ \longrightarrow S^0 \longrightarrow \widetilde{E\mathcal{F}}$$

A point-set model for $E\mathcal{F}$ is $S(\infty\rho)$. ρ = reg. rep. of C_2.

i.e. $E\mathcal{F} = S(\infty\rho) = \bigcup_{n\geq 1} S(n\rho)$ union of unit sphere in $n\rho$.

$$\widetilde{E\mathcal{F}} = S^{\infty\rho} = \operatorname{colim} S^{n\rho}$$

$$= \operatorname{colim}\,(S^0 \xrightarrow{a_\rho} S^\rho \xrightarrow{a_\rho} S^{2\rho} \xrightarrow{a_\rho} \cdots)$$

$$= S^0[a_\rho^{-1}]$$

$\Rightarrow$

$$\begin{array}{ccc} KR \wedge S^0 & \xrightarrow{1\wedge a_\sigma} & KR \wedge S^\rho \\ & \searrow^{\mathrm{id}\wedge\Sigma^{-1}\eta} & \uparrow \beta \\ & & KR \wedge S^{-1} \end{array}$$

where $\eta: S^1 \longrightarrow S^0$ Hopf map

$\beta: S^\rho \longrightarrow KR$ Bott elet. invertible.

$\eta^4 = 0 \quad \Rightarrow$ colim inverts a nilpotent elet η.

- We prove (*) :

Consider the Tate diagram

$$\begin{array}{ccccc} X_{hG} & \longrightarrow & X^G & \longrightarrow & \Phi^G X \\ \simeq \downarrow & & \downarrow & & \downarrow \\ X_{hG} & \xrightarrow{\text{norm}} & X^{hG} & \longrightarrow & X^{tG} \end{array}$$

If $\widetilde{EG} \wedge X$ contractible $\Rightarrow$ $\Phi^G X \simeq *$. X^{tG} is a module over $*$ $\Rightarrow$ $X^{tG} \simeq *$. So $X^G \simeq X^{hG}$.

FACT 4 Homotopy fixed point spectral sequence (HFPSS).

Recall that HFPSS is

$$E_2^{s,t} = H^s(G;\ \pi_t X) \Rightarrow \pi_{t-s}(X^{hG}).$$

This can be obtained by skeleton filtration of $\widetilde{EG}$.

Let $G = C_2$. $X = KR$. π_* replaced by $\underline{\pi}_*^{C_2}$. So we get

$$E_2^{s,t} = H^s(C_2;\ \underline{\pi}_t^{C_2} KR) \Rightarrow \underline{\pi}_{t-s}^{C_2}(KR^{hC_2})$$

$$= \underline{\pi}_{t-s}^{C_2}(KR^{C_2})$$

- Evaluate at C_2/e. $\underline{\pi}_t^{C_2} KR(C_2/e) = \pi_t KU$

$$\underline{\pi}_{t-s}^{C_2}(KR^{C_2})(C_2/e) = \pi_{t-s} KO$$

Let $\beta = H - 1 \in K(S^2)$ be the Bott elt. This β is the underlying map of $S^\rho \to KR$. Note that $\pm 1 \in C_2$ acts on KU by Adams operation. i.e.

$$\begin{array}{ccc} S^{2n,n} & \xrightarrow{\beta^n} & \mathbb{Z} \times BU \\ (-1)^n \downarrow & & \downarrow \psi^{-1} \\ S^{2n,n} & \xrightarrow{\beta^n} & \mathbb{Z} \times BU \end{array}$$

This is b/c $K(S^{2n,n}) \xrightarrow{\psi^{-1}} K(S^{2n,n})$, $\beta^n \in K(S^{2n,n})$ as

$a \longmapsto (-1)^n \cdot a \qquad a$ gen.

Therefore. $\pi_{2t} KU = \mathbb{Z}\{\beta^t\}$, w/ C_2 acts by ψ^{-1}, where $\psi^{-1}\beta^t = (-1)^t \beta^t$. In other word, we get

$$\pi_{2t} KU = \begin{cases} \mathbb{Z}, & t \text{ even} \leftrightarrow \text{trivial action} \\ \mathbb{Z}_-, & t \text{ odd} \leftrightarrow \text{sign rep.} \end{cases}$$

So $H^s(C_2 ; \pi_{2t} KU) \Rightarrow \pi_{2t-s} KO$. need to compute

① $H^s(C_2 ; \mathbb{Z}) = H^s(BC_2 : \mathbb{Z})$

t even $\qquad = H^s(\mathbb{RP}^\infty ; \mathbb{Z})$

$$= \begin{cases} \mathbb{Z}, & s = 0 \\ \mathbb{Z}/2, & s > 0 \text{ even} \\ 0, & \text{else} \end{cases}$$

② To compute $H^s(C_2 : \mathbb{Z}_-)$. note $\mathbb{Z}_- = \mathbb{Z} \curvearrowleft C_2$.

one needs to find the free resolution $P_\bullet \to \mathbb{Z}$ as a trivial $\mathbb{Z}[C_2]$-module: ($\mathbb{Z}[C_2] \cong \mathbb{Z}[x]/x^2-1$)

$$\cdots \quad \mathbb{Z}[C_2] \xrightarrow{(x-1)} \mathbb{Z}[C_2] \xrightarrow{(x+1)} \mathbb{Z}[C_2] \xrightarrow{(x-1)} \mathbb{Z}[C_2] \xrightarrow{x \mapsto 1} \mathbb{Z} \to 0$$

apply $\mathrm{Hom}_{\mathbb{Z}[C_2]}(-, \mathbb{Z}_-)$ to get

$$\mathbb{Z} \xrightarrow{-2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{-2} \mathbb{Z} \xrightarrow{0} \cdots$$

Here we note that $(\mathrm{Hom}_{\mathbb{Z}[C_2]}(-, \mathbb{Z}_-))(x-1) = (-x-1)|_{x=1}$

$$= -2$$

$$(\mathrm{Hom}_{\mathbb{Z}[C_2]}(-, \mathbb{Z}_-))(x+1) = (-x+1)|_{x=1}$$

$$= 0$$

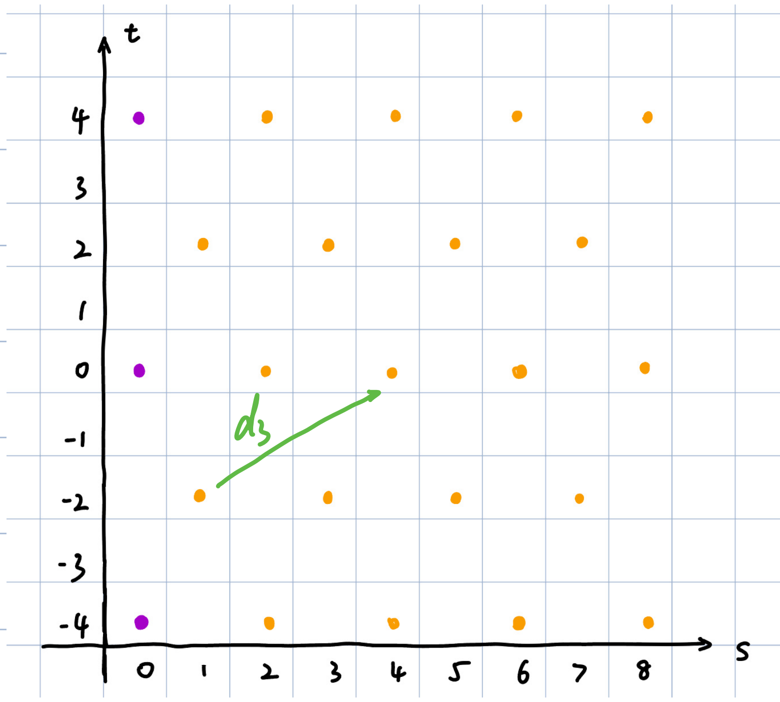
$$\Rightarrow \quad H^s(C_2;\mathbb{Z}_-) = \begin{cases} \mathbb{Z}/2, & s>0,\ \text{odd}. \\ 0, & \text{else}. \end{cases}$$

t odd

Combine ① ②, get

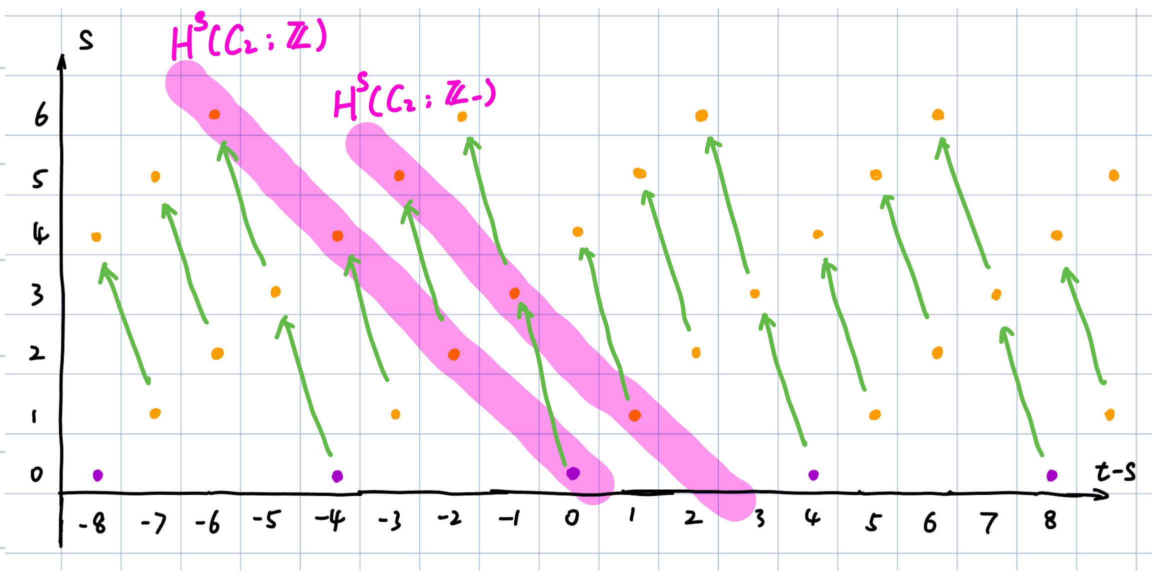
$$E_2^{s,2t} = H^s(C_2;\pi_{2t}KU) = \begin{cases} \mathbb{Z}, & s=0,\ t \text{ even} \\ \mathbb{Z}/2, & t-s \text{ even},\ s>0 \\ 0, & \text{else} \end{cases}$$

E_2 - page : ● = $\mathbb{Z}$ ● = $\mathbb{Z}/2$ in usual grading (s, t)



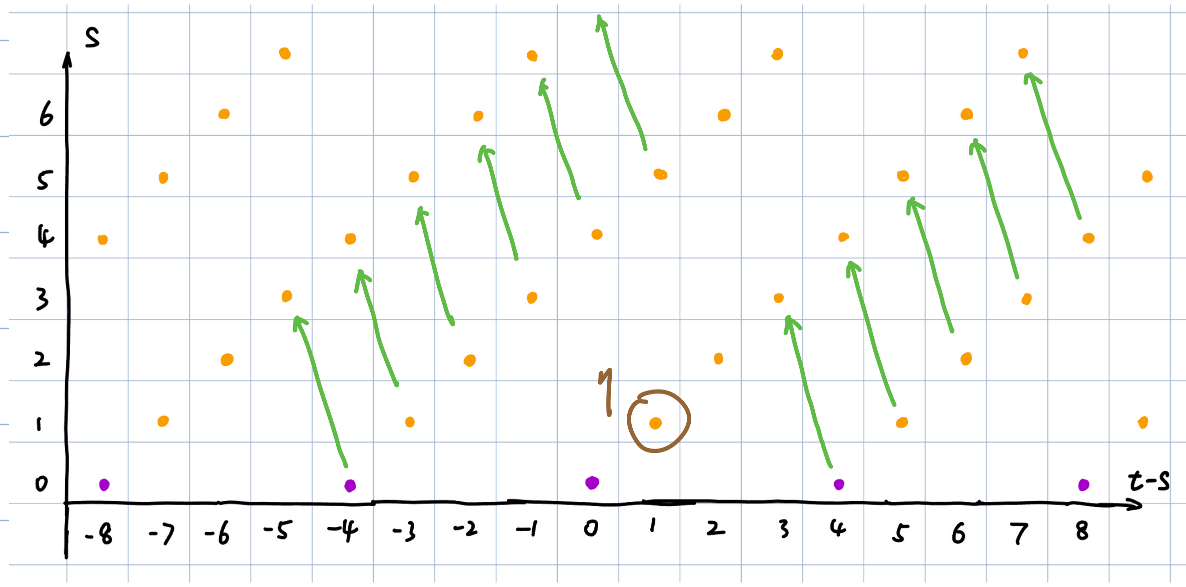
Clearly no d_2 - differentials. First non-trivial diff should be d_3.

E_2 - page in Adams grading $(t-s, s)$: ● = $\mathbb{Z}$ ● = $\mathbb{Z}/2$



Since $\pi_{t-s} KO = \begin{cases} \mathbb{Z}, & t-s \equiv 0, 4 \mod 8 \\ \mathbb{Z}/2, & t-s \equiv 1, 2 \mod 8 \\ 0, & \text{else} \end{cases}$

it follows that some d_3 needs to vanish. Actually we have



Actually one has $\pi_* KO = \mathbb{Z}[\beta^{\pm}, \eta, w] / (2\eta, \eta^3, w^2 = 4\beta)$

where η = Hopf map $S^1 \to S^0$

FACT 5 Slice spectral sequence.

$$E_2^{s,t} = \underline{\pi}_{t-s}^{C_2} P_t KR \Rightarrow \underline{\pi}_{t-s}^{C_2} KR$$

Evaluate at C_2/H, get

$$\underline{\pi}_{t-s}^{C_2} P_t KR(C_2/H) \Rightarrow \underline{\pi}_{t-s}^{C_2} KR(C_2/H) = \begin{cases} \pi_{t-s} KO, & H = C_2 \\ \pi_{t-s} KU, & H = e \end{cases}$$

Need to know the slices. Recall that in Dugger's paper, there is an equivariant Postnikov tower:

$$
\begin{array}{ccc}
\vdots & & \\
\downarrow & & \\
\Sigma^{2,1} kr & \longrightarrow & \Sigma^{2,1} H\mathbb{Z} \\
\downarrow \beta & & \\
kr & \longrightarrow & H\mathbb{Z} \\
\downarrow \Sigma^{-2,-1}\beta & & \\
\Sigma^{-2,-1} kr & \longrightarrow & \Sigma^{-2,-1} H\mathbb{Z} \\
\downarrow & & \\
\vdots & &
\end{array}
$$

where kr = connective cover of KR. β = Bott elt. colim = KR. lim = $*$. So

$$P_t KR = \begin{cases} \Sigma^{t,\frac{t}{2}} H\mathbb{Z} = \Sigma^{\frac{t}{2}\rho} H\mathbb{Z}, & t \text{ even}. \\ * , & t \text{ odd} \end{cases}$$

$$\pi^{C_2}_{2t-s} \Sigma^{t\rho} H\mathbb{Z} = \pi^{C_2}_{2t-s} \Sigma^{t+t\sigma} H\mathbb{Z}$$

$$= \pi^{C_2}_{t-s} (S^{t\sigma} \wedge H\mathbb{Z})$$

$$\cong \begin{cases} H^{C_2}_{t-s}(S^{t\sigma}; \mathbb{Z}), & t \geq 0 \\ H^{t-s}_{C_2}(S^{-t\sigma}; \mathbb{Z}), & t < 0 \end{cases}$$

by S-W dual

Note $H^{t-s}_{C_2}(S^{-t\sigma}; \mathbb{Z}) \cong H^{t-s}(S^{-t\sigma}/C_2 ; \mathbb{Z})$

(Recall $H^*_G(X; \mathbb{Z}) \cong H^*(X/G ; \mathbb{Z})$)

$$= H^{t-s}(S^{V'}/C_2 ; \mathbb{Z})$$

Since $S^{V'} = S^{-t\sigma} = S^{-t,-t}$ = suspension of sphere in $\mathbb{R}^{-t,-t}$

i.e. $(-t-1)$-sphere w/ antipodal action

there is a cofib seq $S(V') \longrightarrow D(V') \longrightarrow S^{-t,-t}$

$$D(V') \cong \mathbb{R}^{-t,-t}, \quad S^{-t,-t} \cong \Sigma S(V')$$

after passing to $H^{t-s}(-) \Rightarrow H^{t-s}(\Sigma S(V')) \cong H^{t-s}(S^{-t,-t})$

$$\Rightarrow \cdots = H^{t-s}(\Sigma \mathbb{RP}^{-t-1} ; \mathbb{Z})$$

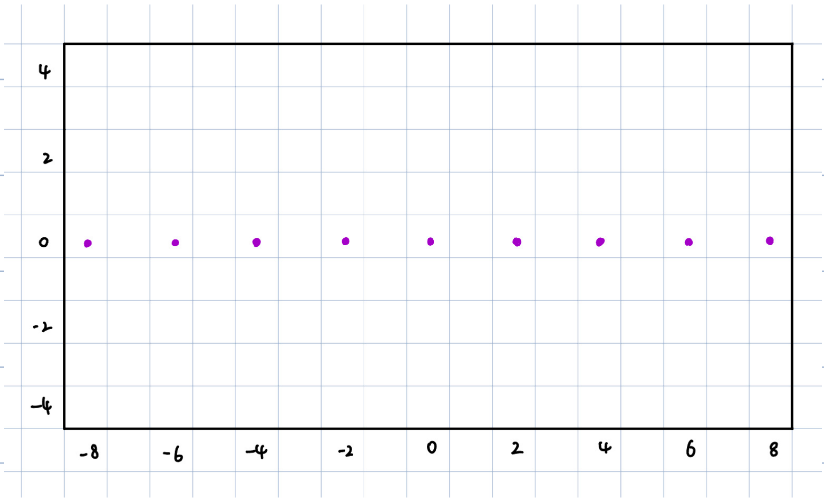
$= H^{t-s-1}(\mathbb{RP}^{-t-1}; \mathbb{Z})$

(also notice $\mathbb{Z}$ is constant)

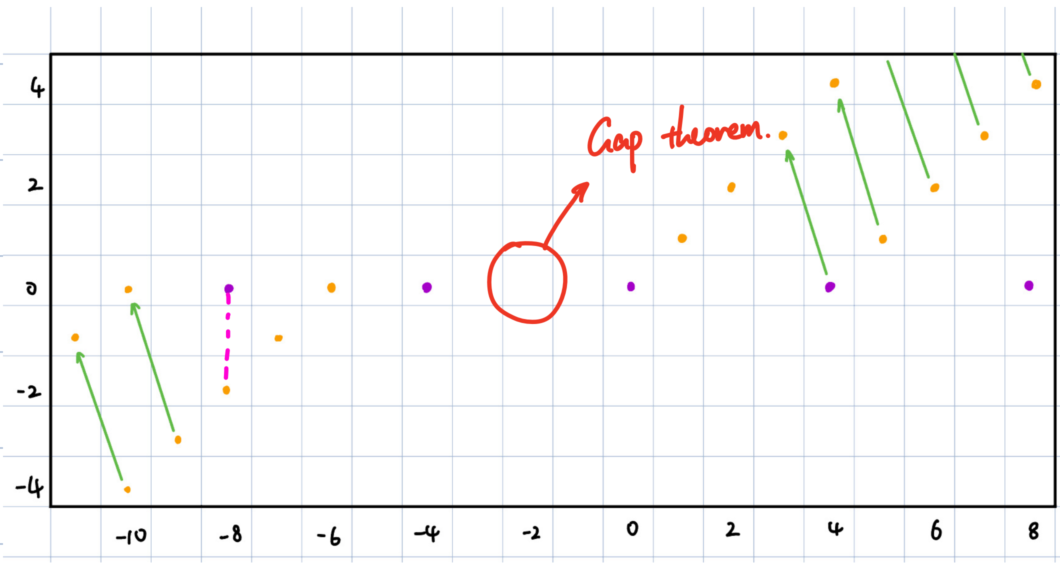
$$= \begin{cases} \mathbb{Z}, & t-s=1 \text{ or } 2t=s,\ t \text{ even} \\ \mathbb{Z}/2, & t-s \text{ odd},\ 1 < t-s < -t,\ 2t=s,\ t \text{ odd} \\ 0, & \text{else} \end{cases}$$

Since we know $\pi_* KU$ and $\pi_* KO$, the differentials are fixed.

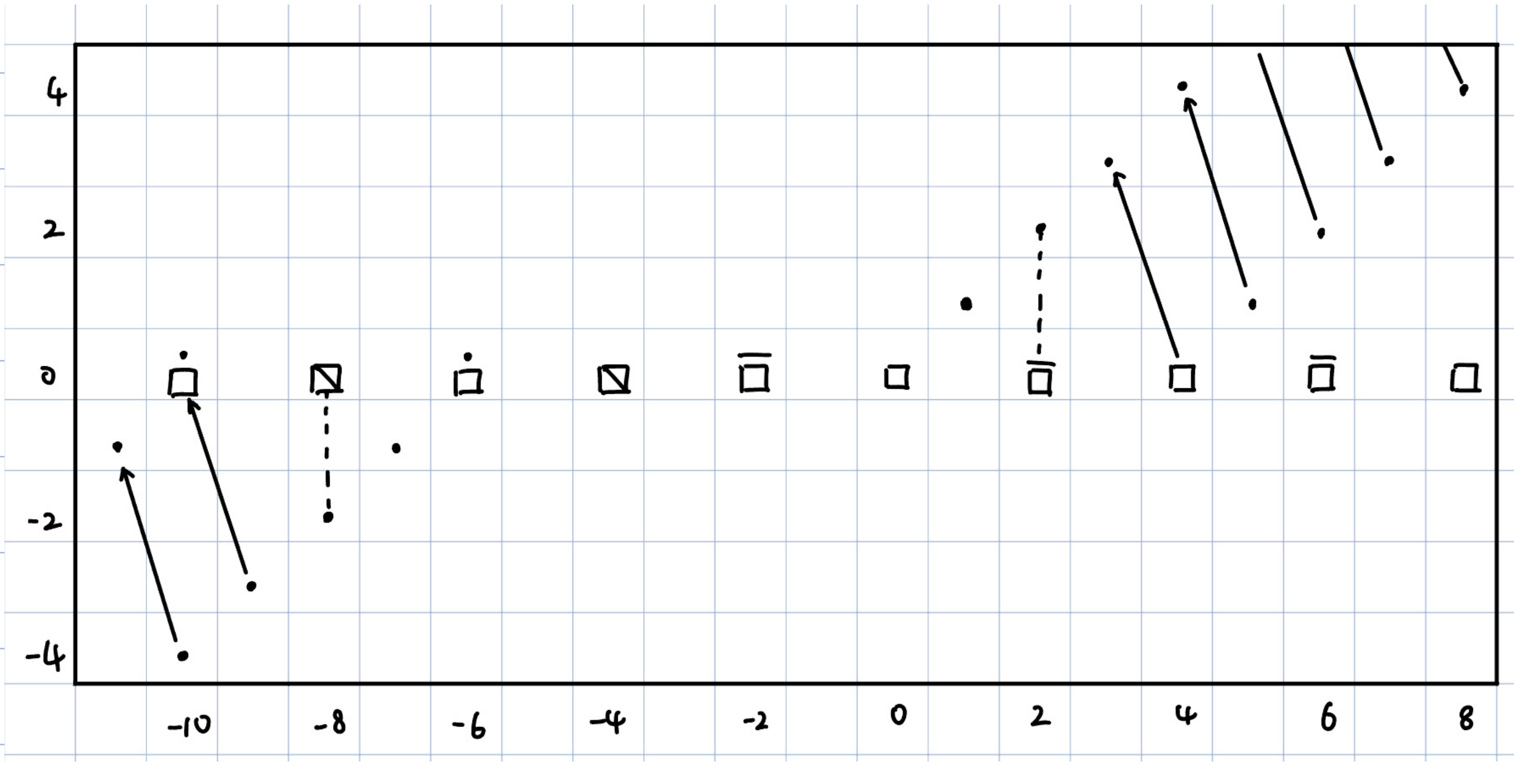
We have, at $H = e$, ($\bullet = \mathbb{Z}$)



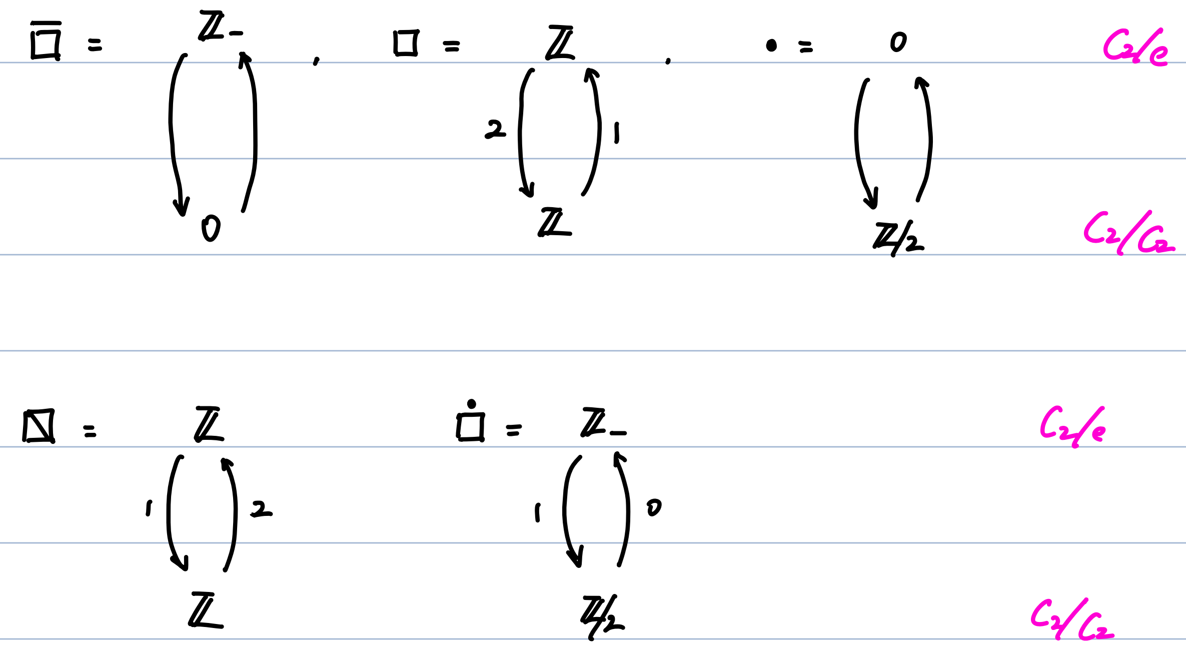
at $H = C_2$, ($\bullet = \mathbb{Z}$, $\bullet = \mathbb{Z}/2$)



Package into Mackey functors together, get :



Here



Thm (Algebraic Gap theorem)

$G = C_2^n$. $V \in R(G)$. then $\underline{\tilde{H}}^*(S^V; \underline{\mathbb{Z}}) = \underline{0}$

for $* = 0, 1$.